Journal of Computational Physics **148,** 397–415 (1999)
Article ID jcph.1998.6120, available online at http://www.idealibrary.com on IDEAL®

Difference Schemes for Solving the Generalized Nonlinear Schrödinger Equation[1]

Qianshun Chang,* Erhui Jia,* and W. Sun†

*Institute of Applied Mathematics, Chinese Academy of Sciences, Beijing 100080, China; and
†Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong

Received April 23, 1998; revised October 1, 1998

This paper studies finite difference schemes for solving the generalized nonlinear Schrödinger (GNLS) equation $iu_t - u_x x + q(|u|^2)u = f(x,t)u$. A new linearlized Crank–Nicolson-type scheme is presented by applying an extrapolation technique to the real coefficient of the nonlinear term in the GNLS equation. Several schemes, including Crank–Nicolson-type schemes, Hopscotch-type schemes, split step Fourier scheme, and pseudospectral scheme, are adopted for solving three model problems of GNLS equation which arise from many physical problems. with $q(s) = s^2$, $q(s) = \ln(1+s)$, and $q(s) = -4s/(1+s)$, respectively. The numerical results demonstrate that the linearized Crank–Nicolson scheme is efficient and robust.

Key Words: difference schemes; generalized Schrödinger equation; linearized Crank–Nicolson scheme.

1. INTRODUCTION

The nonlinear Schrödinger equation (NLS) describes many physical phenomena and has important applications in fluid dynamics, nonlinear optics, and plasma physics. The NLS equation has been investigated analytically and numerically by many authors. For example, Taha and Ablowitz compared eight numerical methods for the NLS in [11].

We consider the generalized nonlinear Schrödinger (GNLS) equation

$$iu_t - u_{xx} + q(|u|^2)u = 0,$$

which arises in plasma physics; see, for example, [1, 2, 6, 7]. In the GNLS equation, the nonlinear term $|u|^2u$ of the NLS is extended to the general form $q(|u|^2)u$, and the function

[1] This work was supported in part by the National Science Foundation of China and City University of Hong Kong Research Grants 7000704 and 7000727.

0021-9991/99 $30.00

$q(s)$ can be chosen as $q(s) = s^p$, $p > 0$, $q(s) = c(1 - e^{-s})$, $q(s) = \frac{s}{1+s}$, or $q(s) = \ln(1+s)$ in different physical problems. Numerical methods for the GNLS equation are studied in [4, 5, 8, 9]. In [8], a pseudospectral solution of the GNLS equation is considered. Conservative difference schemes for the GNLS equation are presented in [4, 5]. In this paper, we consider the initial-boundary value problem of the GNLS equation,

$$iu_t - u_{xx} + q(|u|^2)u = f(x,t)u, \qquad x \in [x_L, x_R], t > 0, \tag{1.1}$$

$$u|_{x=x_L} = 0, \qquad u|_{x=x_R} = 0, \tag{1.2}$$

$$u|_{t=0} = u_0(x), \tag{1.3}$$

where $f(x,t)$ is a real function. We will present several difference schemes for the GNLS equation in (1.1)–(1.3) and evaluate their efficiency.

The initial-boundary value problem (1.1)–(1.3) satisfies the two conservative laws

$$H = \|u(\cdot,t)\|_{L_2}^2 = \|u(\cdot,0)\|_{L_2}^2, \tag{1.4}$$

and

$$E(t) = E(0) + \int_0^t \int_{x_L}^{x_R} f(x,\tau) \cdot \frac{\partial}{\partial \tau}(|u(x,\tau)|^2)\,dx\,d\tau, \tag{1.5}$$

where

$$E(t) = \|u_x(\cdot,t)\|_{L_2}^2 + \int_{x_L}^{x_R} Q(\|u(x,t)\|^2)\,dx, \tag{1.6}$$

and

$$Q(S) = \int_0^S q(Z)\,dZ.$$

A conservative difference scheme for (1.1)–(1.3) proposed in [5] is given by

$$\begin{aligned} &i\left(U_j^{n+1}\right)_{\bar{t}} - \frac{1}{2}\left(\left(U_j^{n+1}\right)_{x\bar{x}} + \left(U_j^n\right)_{x\bar{x}}\right) + \frac{1}{2}\frac{Q\left(\left|U_j^{n+1}\right|^2\right) - Q\left(\left|U_j^n\right|^2\right)}{\left|U_j^{n+1}\right|^2 - \left|U_j^n\right|^2}\left(U_j^{n+1} + U_j^n\right) \\ &\quad = \frac{1}{2} f_j^{n+1/2}\left(U_j^{n+1} + U_j^n\right), \qquad 1 \le j \le J-1,\ n = 0, 1, 2, \ldots, \end{aligned} \tag{1.7}$$

$$U_0^n = U_J^n = 0, \tag{1.8}$$

$$U_j^0 = U_0(x_j), \tag{1.9}$$

where

$$\left(U_j^{n+1}\right)_{\bar{t}} = \frac{1}{\tau}\left(U_j^{n+1} - U_j^n\right), \qquad \left(U_j^n\right)_x = \frac{1}{h}\left(U_{j+1}^n - U_j^n\right), \qquad \left(U_j^n\right)_{\bar{x}} = \frac{1}{h}\left(U_j^n - U_{j-1}^n\right)$$

and $h = (x_R - x_L)/J$ and τ are the space step size and time step size, respectively. It is easy to obtain two discrete conservative laws for the difference problem (1.7)–(1.9), namely,

$$H_h = h\sum_{j=1}^{J-1}\left|U_j^n\right|^2 = h\sum_{j=1}^{J-1}\left|U_j^0\right|^2 \tag{1.10}$$

and

$$E_h^n = E_h^0 + h\tau \sum_{k=0}^{n-1} \sum_{j=1}^{J-1} f_j^{k+1/2} \cdot \left(\left|U_j^{k+1}\right|^2\right)_{\bar{t}}, \tag{1.11}$$

where

$$E_h^n = h \sum_{j=1}^{J-1} \left|\left(U_j^n\right)_x\right|^2 + h \sum_{j=1}^{J-1} Q\left(\left|U_j^n\right|^2\right). \tag{1.12}$$

Comparing (1.10)–(1.12) with (1.4)–(1.6), we see that the difference scheme (1.7)–(1.9) conserves the two invariants that the differential problem (1.1)–(1.3) possesses.

In Ref. [5], the scheme (1.7)–(1.9) was studied carefully. In particular, it was proved in [5] theoretically that under certain conditions, there exists a unique generalized solution of the problem (1.1)–(1.3); the scheme (1.7)–(1.9) is stable and its solution converges to the unique solution of (1.1)–(1.3). In practice, however, the conservative scheme is difficult to use when the nonlinear term $q(s)$ in the GNLS equation is complicated, such as $q(s) = \ln(1+s)$. Moreover, the calculation for the ratio

$$\frac{Q\left(\left|U_j^{n+1}\right|^2 - Q\left(U_j^n\right|^2\right)}{\left|U_j^{n+1}\right|^2 - \left|U_j^n\right|^2}$$

may also be difficult if $|U_j^{n+1}|^2 - |U_j^n|^2$ is small. Therefore, some other difference schemes may be more useful in practice.

The purpose of this work is to investigate eight finite difference schemes for solving the GNLS equation and evaluate their performance. Some of these schemes have been used to solve NLS in [6, 11] and we apply them to the GNLS equation here. We also introduce a new linearized Crank–Nicolson (C-N) scheme in which the extrapolation formula is applied only to the real coefficient of the nonlinear term. In solving nonlinear differential equations, implicit schemes are often required in order to ensure the stability. In an implicit scheme, a system of nonlinear equations is solved at each time step. For this reason, implicit schemes are often very costly to compute. Explicit schemes can be constructed by using extrapolation and hopscotch techniques which will be emphasized in this paper. The former is used to approximate the nonlinear term and the latter is used to discretize the diffusion term. In this paper, these finite difference methods are used to solve the GNLS equation in three model problems: one-soliton solution and two-soliton solution for the cubic Schrödinger equation, plane wave solution for the GNLS equation, and one-soliton solution for the GNLS equation. In view of the results of the numerical experiments, we make the conclusion that the linearized C-N scheme is more efficient and more robust than the other seven schemes, in general.

This paper is organized as follows. In the next section, we describe three model problems of the GNLS equation. Eight finite difference schemes for solving the GNLS equation will be introduced in Section 3, and discussions and analysis on the schemes will be given in Section 4. In Section 5, we will present numerical results and conclusions.

2. MODEL PROBLEM

We consider the finite difference solution of three model problems of the GNLS equation and introduce eight difference schemes, which will be examined numerically in Section 5.

2.1. *The Cubic Schrödinger Equation*

The cubic Schrödinger equation is a basic GNLS equation, in which $q(s) = s$.

(i) One-soliton solution. First, we consider the initial-value problem

$$iu_t - u_{xx} - 2|u|^2 u = 0, \qquad t > 0, \tag{2.1}$$

$$u|_{t=0} = u_0(x) = \operatorname{sech}(x+2) \cdot \exp[-2i(x+2)]. \tag{2.2}$$

The exact solution of (2.1)–(2.2) is

$$u(x,t) = \operatorname{sech}(x + 2 - 4t) \cdot \exp[-i(2x + 4 - 3t)]. \tag{2.3}$$

In our numerical calculation, two boundary conditions are added to (2.1)–(2.2),

$$u|_{x=x_L} = 0, \qquad u|_{x=x_R} = 0, \tag{2.4}$$

where x_L and x_R are chosen to be large enough so that the solution of (2.1), (2.2), and (2.4) approximately agrees with (2.3). Here, we use $x_L = -15$ and $x_R = 15$, and the soliton solution will be computed from $t = 0$ to $t = 1$.

(ii) Collision of two solitons. Second, we consider interacting solitons for the cubic Schrödinger equation (2.1) with initial condition

$$u|_{t=0} = u_0(x) = \operatorname{sech}(x - 10)\exp[-i(2x - 20)] + \operatorname{sech}(x + 10)\exp[i(2x + 20)]. \tag{2.5}$$

The exact solution of the initial-value problem (2.1) and (2.5) is

$$\begin{aligned} u(x,t) = {} & \operatorname{sech}(x - 10 - 4t)\exp[-i(2x - 20 - 3t)] \\ & + \operatorname{sech}(x + 10 + 4t)\exp[i(2x + 20 + 3t)]. \end{aligned} \tag{2.6}$$

The solution includes two solitary waves, which move in the opposite directions. Theoretically, the two solitary waves should emerge from their interaction with their shapes and velocities unchanged [13]. Many numerical results are consistent with the conclusion [8].

In our computation, we add the boundary condition (2.4) to the initial problem of (2.1) and (2.5), and compute this problem in the time interval [0, 1]. In general, one chooses x_L and x_R larger than those in the last problem (2.1)–(2.2) in order to make the boundary condition reasonable. $x_L = -20$ and $x_R = 20$ will be used in our numerical study.

2.2. *Plane Wave Soliton for the GNLS Equation*

The next problem to be solved is the periodic initial-value problem of the GNLS equation:

$$iu_t - u_{xx} + q(|u|^2)u = 0, \tag{2.7}$$

$$u|_{t=0} = u_0(x), \tag{2.8}$$

$$u(x + L) = u(x). \tag{2.9}$$

The problem (2.1)–(2.2) admits a progressive plane wave solution

$$u(x,t) = A \cdot \exp[i(kx - \omega t)]. \tag{2.10}$$

Substitution of (2.10) into (2.7)–(2.9) implies that

$$u|_{t=0} = A \cdot \exp(ikx), \tag{2.11}$$

and

$$\omega + k^2 + q(A^2) = 0. \tag{2.12}$$

We take $A = 2$, $k = \pi$, and $L = 2$. Then

$$\omega = -\pi^2 - q(4),$$

and the velocity of the plane wave is $v = \frac{|\omega|}{\pi}$. We compute the plane wave in the time interval $[0, T]$, $T = \frac{2\pi}{|\omega|}$, during which the wave travels over one period $L = 2$.

2.3. *One-Soliton Solution for the GNLS Equation*

Finally, we consider a more general problem:

$$iu_t - u_{xx} + q(|u|^2)u = f(x,t)u, \tag{2.13}$$

$$u|_{x=x_L} = 0, \qquad u|_{x_R} = 0, \tag{2.14}$$

$$u|_{t=0} = u_0(x). \tag{2.15}$$

The right-hand side $f(x,t)$ can be chosen such that the exact solution is

$$u(x,t) = \exp[-(x - ct)^2 + i(kx - \omega t)]. \tag{2.16}$$

Substituting (2.16) into (2.13) and (2.15) gives

$$f(x,t) = \omega + 2ci(x - ct) - 4(x - ct)^2 + 4ik(x - ct) + k^2 + 2 + q\left(e^{-2(x-ct)^2}\right) \tag{2.17}$$

and

$$u_0(x) = \exp(-x^2 + ikx). \tag{2.18}$$

We take $x_L = -15$, $x_R = 15$, $c = 2$, $k = -1$, and $\omega = -3$. Then,

$$f(x,t) = -4(x - ct)^2 + q\left(e^{-2(x-ct)^2}\right). \tag{2.19}$$

The velocity of the soliton solution is 2. The soliton wave will be computed from $t = 0$ to $t = 5$.

3. FINITE DIFFERENCE SCHEMES

In this section, we describe eight difference schemes for the GNLS equation (1.1) with application to the three model problems described in the previous section. Some of these schemes have been used to solve the NLS equation in [11].

3.1. *The Crank–Nicolson Implicit Scheme*

The first difference scheme to be considered is the well-known implicit C-N scheme, which is given by

$$
\begin{aligned}
i\left(U_j^{n+1}\right)_{\bar{t}} - \frac{1}{2}\left(\left(U_j^{n+1}\right)_{x\bar{x}} + \left(U_j^{n}\right)_{x\bar{x}}\right) + \frac{1}{2}q\left(\frac{\left|U_j^{n+1}\right|^2 + \left|U_j^{n}\right|^2}{2}\right)\left(U_j^{n+1} + U_j^{n}\right) \\
= \frac{1}{2} f_j^{n+1/2}\left(U_j^{n+1} + U_j^{n}\right). \qquad (3.1)
\end{aligned}
$$

This scheme is equivalent to the conservative scheme (1.7) when applied to the cubic Schrödinger equation since $q(s) = s$ and $Q(s) = s^2/2$. However, (3.1) is more convenient for implementation. The truncation error of this scheme is of order $O(\tau^2 + h^2)$. According to linearized stability analysis, this scheme is unconditionally stable.

A nonlinear iterative algorithm can be used to solve the system of the nonlinear equation (3.1). The iterative algorithm for Eq. (3.1) can be written as

$$
\begin{aligned}
-\frac{\gamma}{2}\left(U_{j+1}^{n+1}\right)^{s+1} + \left[i + \gamma + \frac{\tau}{2}q\left(\frac{\left|\left(U_j^{n+1}\right)^s\right|^2 + \left|U_j^n\right|^2}{2}\right) - \frac{\tau}{2} f_j^{n+1/2}\right]\left(U_j^{n+1}\right)^{s+1} \\
-\frac{\gamma}{2}\left(U_{j-1}^{n+1}\right)^{s+1} = iU_j^n + \frac{\gamma}{2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right) \\
-\frac{\tau}{2}\left[q\left(\frac{\left|\left(U_j^{n+1}\right)^s\right|^2 + \left|U_j^n\right|^2}{2}\right) - f_j^{n+1/2}\right]U_j^n,
\end{aligned}
$$

where $\gamma = \tau/h^2$. The superscript s denotes the sth iterate for solving the nonlinear difference equations at each time step. The initial iterate $(U_j^{n+1})^0$ is chosen as

$$
\left(U_j^{n+1}\right)^0 = U_j^n.
$$

In each iteration, a tridiagonal system of equations can be solved by Gaussian elimination method. The iteration continues until the condition

$$
\max_j \left|\left(U_j^{n+1}\right)^{s+1} - \left(U_j^{n+1}\right)^s\right| < 10^{-6}
$$

is satisfied, and the value $(U_j^{n+1})^{s+1}$ is used as U_j^{n+1}. The iteration procedure is repeated at each time level.

3.2. *A Three-Level Explicit Scheme*

This is a classical explicit scheme with central difference in time for stability. The scheme for GNLS (1.1) is

$$i\,\frac{U_j^{n+1}-U_j^{n-1}}{2\tau}-\left(U_j^n\right)_{x\bar{x}}+q\left(\left|U_j^n\right|^2\right)U_j^n = f_j^n\cdot U_j^n. \tag{3.2}$$

The truncation error of this scheme is of order $O(\tau^2+h^2)$. According to a linearized stability analysis, this scheme is stable if $\gamma \le \frac{1}{4}$.

3.3. *A Hopscotch Scheme*

The GNLS equation (1.1) can be approximated in two steps: an explicit step at the odd values of $(n+j)$,

$$i\left(U_j^{n+1}\right)_{\bar{t}}-\left(U_j^n\right)_{x\bar{x}}+\frac{1}{2}\left[q\left(\left|U_{j+1}^n\right|^2\right)U_{j+1}^n+q\left(\left|U_{j-1}^n\right|^2\right)U_{j-1}^n\right] = f_j^n\cdot U_j^n, \tag{3.3}$$

and an implicit step at the even values of $(n+j)$,

$$i\left(U_j^{n+1}\right)_{\bar{t}}-\left(U_j^{n+1}\right)_{x\bar{x}}+\frac{1}{2}\left[q\left(\left|U_{j+1}^{n+1}\right|^2\right)U_{j+1}^{n+1}+q\left(\left|U_{j-1}^{n+1}\right|^2\right)U_{j-1}^{n+1}\right] = f_j^{n+1}\cdot U_j^{n+1}. \tag{3.4}$$

Here, U_j^{n+1} at the odd values of $(n+j)$ can be calculated by (3.3) and then U_j^{n+1} at the even values of $(n+j)$ by (3.4). Therefore, (3.4) becomes an explicit formula. Combining (3.3) with (3.4), Eq. (3.3) may be replaced by the extrapolation formula

$$U_j^{n+1} = 2U_j^n - U_j^{n-1}, \qquad \textit{for odd values of } (n+j). \tag{3.5}$$

This scheme is unconditionally stable by means of linearized stability analysis. Its truncation error will be analyzed in Section 4.

3.4. *Linearized Crank–Nicolson Scheme I*

In the C-N implicit scheme (3.1), one has to solve the nonlinear difference equations iteratively at each time step. This is time consuming in general. In order to avoid the iterative process, we consider an extrapolation formula to approximate the nonlinear term and obtain the linearized C-N scheme

$$\begin{aligned} i\left(U_j^{n+1}\right)_{\bar{t}}-\frac{1}{2}\left(\left(U_j^{n+1}\right)_{x\bar{x}}+\left(U_j^n\right)_{x\bar{x}}\right)+\frac{3}{2}q\left(\left|U_j^n\right|^2\right)U_j^n-\frac{1}{2}q\left(\left|U_j^{n-1}\right|^2\right)U_j^{n-1}\\ = \frac{1}{2}f_j^{n+1/2}\left(U_j^{n+1}+U_j^n\right). \end{aligned} \tag{3.6}$$

This is a semi-implicit method, because only a linear tridiagonal system of equations needs to be solved at each time level. The truncation error of this scheme is of order $O(\tau^2+h^2)$. The scheme is unconditionally stable by a linearized stability analysis.

In [10], this linearized C-N scheme was used to compute the two-dimensional cubic Schrödinger equation.

3.5. *Linearized Crank–Nicolson Scheme II*

In the above linearized C-N scheme (3.6), the extrapolation formula is used to approximate the entire nonlinear term $q(|U|^2)U$. Next, we present a new linearized C-N scheme. In this scheme, the extrapolation formula is used to approximate only the real coefficient $q(|U|^2)$ of the nonlinear term, and the scheme can be written as

$$i\left(U_j^{n+1}\right)_{\bar{t}} - \frac{1}{2}\left(\left(U_j^{n+1}\right)_{x\bar{x}} + \left(U_j^n\right)_{x\bar{x}}\right) + \frac{1}{2}\left(\frac{3}{2}q\left(\left|U_j^n\right|^2\right) - \frac{1}{2}q\left(\left|U_j^{n-1}\right|^2\right)\right)\left(U_j^{n+1} + U_j^n\right)$$
$$= \frac{1}{2} f_j^{n+1/2}\left(U_j^{n+1} + U_j^n\right). \tag{3.7}$$

The scheme (3.7) is expected to be more efficient than (3.6), since the former uses the extrapolation for the real coefficient of the nonlinear term only. Furthermore, as we will show in the Section 4, the scheme (3.7) satisfies a conservation law. The truncation error of this scheme is also of order $O(\tau^2 + h^2)$. It is unconditionally stable by means of linearized stability analysis.

3.6. *Hopscotch Scheme with Extrapolation*

In this scheme, the hopscotch algorithm is used for the diffusion term and the extrapolation formula is used for the nonlinear term as used in (3.6). Then we have

$$i\left(U_j^{n+1}\right)_{\bar{t}} - \left(U_j^n\right)_{x\bar{x}} + \frac{1}{2}\left(\frac{3}{2}q\left(\left|U_j^n\right|^2\right) - \frac{1}{2}q\left(\left|U_j^{n-1}\right|^2\right)\right)\left(U_j^{n+1} + U_j^n\right)$$
$$= \frac{1}{2} f_j^{n+1/2}\left(U_j^{n+1} + U_j^n\right), \tag{3.8}$$

if $(n + j)$ is odd, and

$$i\left(U_j^{n+1}\right)_{\bar{t}} - \left(U_j^{n+1}\right)_{x\bar{x}} + \frac{1}{2}\left(\frac{3}{2}q\left(\left|U_j^n\right|^2\right) - \frac{1}{2}q\left(\left|U_j^{n-1}\right|^2\right)\right)\left(U_j^{n+1} + U_j^n\right)$$
$$= \frac{1}{2} f_j^{n+\frac{1}{2}}\left(U_j^{n+1} + U_j^n\right), \tag{3.9}$$

if $(n + j)$ is even. This scheme is unconditionally stable in linear stability analysis and its truncation error will be analyzed in Section 4.

3.7. *Pseudospectral Scheme (Fornberg and Whitham)*

Using the idea of Fornberg and Whitham [12], the GNLS equation (1.1) is approximated by

$$U_j^{n+1} = U_j^{n-1} + 2iF_j^{-1}\left(\sin\left(\frac{k^2\pi^2}{P^2}\tau\right)F_k(U^n)\right) + 2i\tau q\left(\left|U_j^n\right|^2\right)U_j^n - 2i\tau f_j^n \cdot U_j^n, \tag{3.10}$$

where P is half the length of the space interval of interest and F_k denotes a discrete Fourier transform. This scheme is unconditionally stable according to linear analysis. Its truncation error is of order $O(\tau^2 + h^m)$, where m depends on the smoothness of the exact solution. We can take m to be any positive number for our test problems in this paper.

3.8. *Split Step Fourier Method*

According to the algorithm in [11], the split step Fourier is given by

$$\tilde{U}(x, t+\tau) = \exp^{(i(q(|u(x,t)|^2) - f(x,t))\tau)} U(x, t), \tag{3.11}$$

$$U_j^{n+1} \equiv U(x_j, t+\tau) = F_j^{-1}\left(\exp^{(ik^2\tau\pi^2/p^2)} F_k(\tilde{U}(x_j, t+\tau))\right). \tag{3.12}$$

This scheme is unconditionally stable by means of linear analysis. Its truncation error is of order $O(\tau^2 + h^m)$, which is the same as the pseudospectral scheme (3.10). It has been noted that the calculation in (3.10) and (3.12) can be performed in terms of FFT.

4. DISCUSSIONS OF THE FINITE DIFFERENCE SCHEMES

In this sections, we discuss some properties of the difference schemes introduced in the previous section.

4.1. *Conservation Properties*

The initial-boundary value problem of the GNLS equation (1.1)–(1.3) satisfies two conservation laws given by (1.4)–(1.6). It is desirable for a finite difference scheme to preserve discrete analogoues of these invariant quantities. We have the following result about the conservation properties.

PROPOSITION. *Three difference schemes of Section 3, the Crank–Nicolson implicit scheme* (3.1), *the three level explicit scheme* (3.2), *and the Crank–Nicolson scheme II* (3.7), *satisfy some discrete conservation laws.*

Specifically, schemes (3.1) *and* (3.7) *satisfy* $H_h = \text{const}$, *where* $H_h = h\sum_{j=1}^{J-1}|U_j^n|^2$, *and scheme* (3.2) *satisfies* $h(U_j^{n+1}, U_j^n) = h\sum_{j=1}^{J-1} U_j^{n+1}\overline{U_j^n} = \text{const}$.

Proof: Let $U_j^{n+1/2} = \frac{1}{2}(U_j^{n+1} + U_j^n)$. Then

$$\begin{aligned} &i\left(\left(U_j^{n+1}\right)_{\bar{t}}, 2U_j^{n+1/2}\right) - \frac{1}{2}\left(\left(U_j^{n+1}\right)_{x\bar{x}} + \left(U_j^n\right)_{x\bar{x}}, 2U_j^{n+1/2}\right) \\ &\quad + \frac{1}{2}\left(q\left(\frac{\left|U_j^{n+1}\right|^2 + \left|U_j^n\right|^2}{2}\right)\left(U_j^{n+1} + U_j^n\right), 2U_j^{n+1/2}\right) \\ &= \frac{1}{2}\left(f_j^{n+1/2}\left(U_j^{n+1} + U_j^n\right), 2U_j^{n+1/2}\right), \end{aligned}$$

or

$$\begin{aligned} &i\left(\left(U_j^{n+1}\right)_{\bar{t}}, 2U_j^{n+1/2}\right) - \frac{1}{2}\left(\left(U_j^{n+1}\right)_{x\bar{x}} + \left(U_j^n\right)_{x\bar{x}}, 2U_j^{n+1/2}\right) \\ &\quad + 2q\left(\frac{\left|U_j^{n+1}\right|^2 + \left|U_j^n\right|^2}{2}\right)\left(U_j^{n+1/2}, U_j^{n+1/2}\right) \\ &= 2f_j^{n+\frac{1}{2}}\left(U_j^{n+1/2}, U_j^{n+1/2}\right). \end{aligned} \tag{4.1}$$

Since $((U_j^{n+1})_{\bar{t}}, 2U_j^{n+1/2}) = \frac{1}{\tau}(U_j^{n+1} - U_j^n, U_j^{n+1} + U_j^n)$, we have

$$\mathrm{Re}\big(\big(U_j^{n+1}\big)_{\bar{t}}, 2U_j^{n+1/2}\big) = \frac{1}{\tau}\,\mathrm{Re}\big(U_j^{n+1} - U_j^n, U_j^{n+1} + U_j^n\big) = \frac{1}{\tau}\Big(\big|U_j^{n+1}\big|^2 - \big|U_j^n\big|^2\Big).$$

Also,

$$\begin{aligned}\frac{1}{2}\big(\big(U_j^{n+1}\big)_{x\bar{x}} + \big(U_j^n\big)_{x\bar{x}}, 2U_j^{n+1/2}\big) &= 2\big(\big(U_j^{n+1/2}\big)_{x\bar{x}}, U_j^{n+1/2}\big)\\ &= -2\big(\big(U_j^{n+1/2}\big)_x, \big(U_j^{n+1/2}\big)_x\big) = -2\big|\big(U_j^{n+1/2}\big)_x\big|^2.\end{aligned}$$

Thus all the terms except the first one in Eq. (4.1) are real. Taking the imaginary part of (4.1) and summing it over j, we obtain

$$\sum_{j=1}^{J-1}\big|U_j^{n+1}\big|^2 = \sum_{j=1}^{J-1}\big|U_j^n\big|^2, \qquad \text{for all } n.$$

This shows that scheme (3.1) satisfies the conservation law $H_h = \text{const}$. Similarly, it can be verified that $H_h = \text{const}$. for scheme (3.7).

Taking inner product of (3.2) with U_j^n, the same procedure as above for scheme (3.2) leads to

$$h\big(U_j^{n+1}, U_j^n\big) = h\sum_{j=1}^{J-1} U_j^{n+1}\overline{U_j^n} = \text{const}.$$

This completes the proof. ■

The linearized C-N scheme I does not satisfy any conservation laws, since the extrapolation is applied to the entire nonlinear term. Schemes (3.3)–(3.4) and (3.8)–(3.9) are not conservative either, since implicit and explicit steps are used alternatively in these schemes. The pseudospectral scheme (3.10) and split step Fourier method (3.11)–(3.12) are not conservative because the higher order derivative term is approximated by discrete transform. None of the eight schemes satisfies the conservation law (1.11)–(1.12) about E_h^n.

4.2. *Comparison of the Linearized C-N Schemes I and II*

In the previous subsection, we have shown that the linearized C-N scheme II admits an invariant H_h, but the linearized C-N scheme I is nonconservative. In the following, we compare the two schemes further by studying plane wave solutions.

The test problem in subsection 2.2 admits plane wave solutions of the form (2.8). The discrete analogue of a plane wave solution is

$$U_j^n = A\lambda^n e^{ikjh}, \tag{4.2}$$

where A, λ, and k are constants independent of n and j. Substituting (4.2) into (3.7) and noting

$$\big(U_j^n\big)_{x\bar{x}} = A\lambda^n e^{ikjh}\frac{1}{h^2}(e^{ikh} - 2 + e^{-ikh}) = -\frac{4}{h^2}U_j^n \sin^2\frac{kh}{2},$$

we obtain

$$\frac{i}{\tau}\left(U_j^{n+1}-U_j^n\right)+\frac{2}{h^2}\left(U_j^{n+1}+U_j^n\right)\sin^2\frac{kh}{2}$$
$$+\frac{1}{2}\left(\frac{3}{2}q\left((A|\lambda|^n)^2\right)-\frac{1}{2}q\left((A|\lambda|^{n-1})^2\right)\right)\left(U_j^{n+1}+U_j^n\right)=0$$

or

$$\frac{i}{\tau}(\lambda-1)+\left(\frac{2}{h^2}\sin^2\frac{kh}{2}+\frac{1}{2}\left(\frac{3}{2}q\left((A|\lambda|^n)^2\right)-\frac{1}{2}q\left((A|\lambda|^{n-1})^2\right)\right)\right)(\lambda+1)=0.$$

Solving the above equation for λ gives

$$\lambda=\frac{i-\frac{2\tau}{h^2}\sin^2\frac{kh}{2}-\frac{\tau}{2}\left(\frac{3}{2}q\left((A|\lambda|^n)^2\right)-\frac{1}{2}q\left((A|\lambda|^{n-1})^2\right)\right)}{i+\frac{2\tau}{h^2}\sin^2\frac{kh}{2}+\frac{\tau}{2}\left(\frac{3}{2}q\left((A|\lambda|^n)^2\right)-\frac{1}{2}q\left((A|\lambda|^{n-1})^2\right)\right)}.$$

Since the numerator and denominator are complex conjugates, we have $|\lambda|=1$. Therefore, the linearized C-N scheme II admits plane wave solutions, and the scheme is nondissipative. Let $\lambda=e^{-i\omega\tau}$. The dispersion relation of the scheme is

$$\omega=-\frac{1}{\tau}\arcsin\frac{\frac{4\tau}{h^2}\sin^2\frac{kh}{2}+\tau q(A^2)}{1+\tau^2\left(\frac{2}{h^2}\sin^2\frac{kh}{2}+\frac{1}{2}q(A^2)\right)^2}=-(k^2+q(A^2))+O(\tau^2).$$

Comparing this with the dispersion relation (2.12) of the GNLS equation, we see that the phase error is $O(\tau^2)$.

For linearized C-N scheme I, substituting (4.2) into (3.6) leads to

$$\lambda^2\left(i+\frac{2\tau}{h^2}\sin^2\frac{kh}{2}\right)-\lambda\left(i-\frac{2\tau}{h^2}\sin^2\frac{kh}{2}-\frac{3\tau}{2}q\left((A|\lambda|^n)^2\right)\right)-\frac{\tau}{2}q\left((A|\lambda|^{n-1})^2\right)=0. \tag{4.3}$$

Clearly, any root λ of Eq. (4.3) is a function of n. Therefore, no plane wave solution exists for the linearized C-N scheme I. To estimate the phase error, we substitute $\lambda=|\lambda|e^{-i\omega\tau}$ into (4.3) and assume $|\lambda|=1+O(\tau^2)$. This results in

$$\lambda=1+i\left(\frac{4\tau}{h^2}\sin^2\frac{kh}{2}+\tau q(A^2)\right)+O(\tau^2).$$

Then, an approximate dispersion relation is

$$\omega=-(k^2+q(A^2))+O(\tau).$$

It is obvious that the phase error in the linearized C-N scheme I is larger than that in the linearized C-N scheme II.

4.3. *Truncation Errors of the Hopscotch Schemes*

By using Taylor expansion and the estimation of the error between the continuous Fourier transform and discrete Fourier transform, it is straightforward to verify that the truncation errors for the six schemes except the hopscotch-type schemes are $O(\tau^2 + h^2)$ or better.

In the hopscotch scheme (3.3)–(3.4), implicit and explicit steps are performed alternatively. For a fixed j, the solution is computed by (3.4) at a time step m, by (3.3) at time step $m+1$, and by (3.4) again at time step $m+2$. At the time step $m+1$, this is equivalent to a three-level scheme given by

$$i\frac{U_j^{m+1} - U_j^{m-1}}{\tau} - 2\left(U_j^m\right)_{x\bar{x}} + q\left(\left|U_{j+1}^m\right|^2\right)U_{j+1}^m + q\left(\left|U_{j-1}^m\right|^2\right)U_{j-1}^m = f_j^m \cdot U_j^m.$$

Its truncation error is $O(\tau^2 + h^2)$. At the time step $m+2$, the scheme is equivalent to

$$\begin{aligned} i\frac{U_j^{m+2} - U_j^m}{\tau} &- \left(\left(U_j^{m+2}\right)_{x\bar{x}} + \left(U_j^m\right)_{x\bar{x}}\right) + \frac{1}{2}\left(q\left(\left|U_{j+1}^{m+2}\right|^2\right)U_{j+1}^{m+2} + q\left(\left|U_{j-1}^{m+2}\right|^2\right)U_{j-1}^{m+2}\right. \\ &\left. + q\left(\left|U_{j+1}^m\right|^2\right)U_{j+1}^m + q\left(\left|U_{j-1}^m\right|^2\right)U_{j-1}^m\right) = f_j^{m+2} \cdot U_j^{m+2} + f_j^m \cdot U_j^m. \end{aligned} \tag{4.4}$$

By Taylor expansion, we see that the truncation error of (4.4) is of order $O(\tau^2 + h^2 + \tau^2/h^2)$, which is also true for many other physical equations. Therefore, the condition $\tau = o(h)$ is required in the hopscotch schemes to ensure the convergence.

5. NUMERICAL EXPERIMENTS AND CONCLUSIONS

In this section, we use the eight difference schemes to compute the three test problems given in the previous section. To compare the results, we use the approach of [11]. In this approach, we fix the accuracy (L_∞) from $t = 0$ to $t = T$, and leave step sizes h and τ free. We then compare the computing time required to attain such accuracy for various choices of the parameters. For each scheme, the step sizes h and τ are chosen such that the scheme is stable and the least computing time is used to attain the given accuracy.

The following notation will be used when presenting the results,

$$\mathcal{V}_1 = \frac{\left(H_h^n - H_h^0\right)}{H_h^0},$$

$$\mathcal{V}_2 = \frac{\left(E_h^n - E_h^0\right)}{E_h^0},$$

$$L_\infty = \max_j \left|U_j^n - u\left(x_j, t^n\right)\right|,$$

and

$$R_\infty = L_\infty \Big/ \max_j \left|u\left(x_j, t^n\right)\right|,$$

where $u(x_j, t^n)$ is the exact solution at the point (x_j, t^n).

The eight schemes are used for all the three-model problems. The computing times for solving the problems to a given accuracy are reported in Tables I–V. In each table, the first

TABLE I

Comparison of the Computing Time Required to Attain an Accuracy $L_\infty < 0.01$ from $t = 0$ to $t = 1$ for the Model Problem 2.1(i)

No.	Method	Step size	Time (s)	$\mathcal{L}_\infty$	$\mathcal{V}_1$	$\mathcal{V}_2$	Ratio
1	C-N scheme	$h = 0.05$ $\tau = 0.008$	9.34	0.0091	0.00000	0.00000	1
2	Explicit scheme	$h = 0.06$ $\tau = 0.0009$	12.84	0.0098	0.00000	0.00000	1.37
3	Hopscotch scheme I	$h = 0.1$ $\tau = 0.003$	1.57	0.0081	0.00003	0.00021	0.17
4	Linearized C-N scheme I	$h = 0.05$ $\tau = 0.005$	5.44	0.0094	0.00002	0.00003	0.58
5	Linearized C-N scheme II	$h = 0.05$ $\tau = 0.007$	3.87	0.0098	0.00000	0.00002	0.41
6	Hopscotch scheme II	$h = 0.05$ $\tau = 0.0005$	20.32	0.0097	0.00000	0.00001	2.18
7	Pseudospectral scheme	$h = 0.15625$ $\tau = 0.004$	3.56	0.0011	0.00008	0.00010	0.38
8	Split step Fourier method	$h = 0.15625$ $\tau = 0.004$	1.13	0.0038	0.00005	0.00007	0.12

TABLE II

Comparison of the Computing Time Required to Attain an Accuracy $L_\infty < 0.01$ from $t = 0$ to $t = 1$ for the Model Problem 2.1(ii)

No.	Method	Step size	Time (s)	$\mathcal{L}_\infty$	$\mathcal{V}_1$	$\mathcal{V}_2$	Ratio
1	C-N scheme	$h = 0.05$ $\tau = 0.008$	13.56	0.0091	0.0000	0.0000	1
2	Explicit scheme	$h = 0.05$ $\tau = 0.0006$	21.57	0.0082	0.00000	0.00000	1.59
3	Hopscotch scheme I	$h = 0.1$ $\tau = 0.003$	2.34	0.0081	0.00003	0.00021	0.17
4	Linearized C-N scheme I	$h = 0.05$ $\tau = 0.005$	8.91	0.0094	0.00002	0.00000	0.66
5	Linearized C-N scheme II	$h = 0.05$ $\tau = 0.007$	6.09	0.0098	0.00000	0.00000	0.45
6	Hopscotch scheme II	$h = 0.04$ $\tau = 0.0004$	62.04	0.0083	0.00001	0.00000	4.58
7	Pseudospectral scheme	$h = 0.15625$ $\tau = 0.002$	8.82	0.0050	0.00003	0.00007	0.65
8	Split step Fourier method	$h = 0.15625$ $\tau = 0.001$	12.78	0.0087	0.00002	0.00003	0.94

TABLE III

Comparison of the Computing Time Required to Attain an Accuracy $L_\infty < 0.02$ for Computations of the Plane Wave from $t = 0$ to $t = \frac{2\pi}{\omega}$ for the Model Problem 2.2

No.	Method	$q(S)$	Step size	Time (s)	L_∞	$\mathcal{V}_1$	$\mathcal{V}_2$	Ratio
1	C-N scheme	S^2	$h = 0.04$ $\tau = 0.004$	0.35	0.0181	0.00000	0.00000	1
		$-\frac{4S}{1+S}$	$h = 0.02$ $\tau = 0.01$	1.09	0.0112	0.00003	0.00003	1
		$\ln(1+S)$	$h = 0.04$ $\tau = 0.005$	0.65	0.0177	0.00001	0.00001	1
2	Explicit scheme	S^2	$h = 0.04$ $\tau = 0.00039$	0.36	0.0060	0.00000	0.00000	1.03
		$-\frac{4S}{1+S}$	$h = 0.025$ $\tau = 0.00015$	5.87	0.0103	0.00000	0.00000	5.39
		$\ln(1+S)$	$h = 0.04$ $\tau = 0.00039$	0.97	0.0142	0.00000	0.00000	1.49
3	Hopscotch scheme I	S^2	$h = 0.02$ $\tau = 0.00005$	7.04	0.0179	0.00004	0.00007	20.11
		$-\frac{4S}{1+S}$	$h = 0.04$ $\tau = 0.0008$	0.92	0.0105	0.00001	0.00001	0.84
		$\ln(1+S)$	$h = 0.02$ $\tau = 0.0001$	8.57	0.0101	0.00016	0.00537	13.18
4	Linearized C-N scheme I	S^2	$h = 0.02$ $\tau = 0.00001$					
		$-\frac{4S}{1+S}$	$h = 0.02$ $\tau = 0.00001$		$L_\infty > 0.4$			
		$\ln(1+S)$	$h = 0.02$ $\tau = 0.00001$					
5	Linearized C-N scheme II	S^2	$h = 0.04$ $\tau = 0.004$	0.17	0.0176	0.00000	0.00000	0.49
		$-\frac{4S}{1+S}$	$h = 0.02$ $\tau = 0.01$	0.53	0.0111	0.00003	0.00003	0.49
		$\ln(1+S)$	$h = 0.04$ $\tau = 0.005$	0.32	0.0177	0.00001	0.00001	0.49
6	Hopscotch scheme II	S^2	$h = 0.04$ $\tau = 0.0002$	1.30	0.0098	0.00004	0.00007	3.71
		$-\frac{4S}{1+S}$	$h = 0.02$ $\tau = 0.0002$	10.28	0.0180	0.00009	0.00008	9.43
		$\ln(1+S)$	$h = 0.04$ $\tau = 0.0002$	3.32	0.0174	0.00001	0.00116	9.22
7	Pseudospectral scheme	S^2	$h = 0.25$ $\tau = 0.003$	0.16	0.0137	0.00106	0.00180	0.46
		$-\frac{4S}{1+S}$	$h = 0.3125$ $\tau = 0.008$	0.54	0.0135	0.00002	0.00002	0.50
		$\ln(1+S)$	$h = 0.3125$ $\tau = 0.001$	1.09	0.000018	0.00001	0.00001	1.68
8	Split step Fourier method	S^2	$h = 0.125$ $\tau = 0.002$	0.53	0.0185	0.00187	0.00235	1.51
		$-\frac{4S}{1+S}$	$h = 0.15625$ $\tau = 0.008$	1.06	0.0174	0.00005	0.00004	0.97
		$\ln(1+S)$	$h = 0.3125$ $\tau = 0.001$	1.14	0.0047	0.00001	0.00001	1.75

TABLE IV

Comparison of the Computing Time Required to Attain an Accuracy $L_\infty < 0.05$ for Computations from $t = 0$ to $t = 5$ for the Model Problem 2.3

No.	Method	$q(S)$	Step size	Time (s)	$Ł_\infty$	$\mathcal{V}_1$	Ratio
1	C-N scheme	S^2	$h = 0.1$ $\tau = 0.01$	13.54	0.0314	0.00000	1
		$-\frac{4S}{1+S}$	$h = 0.1$ $\tau = 0.01$	13.85	0.0490	0.00000	1
		$\ln(1+S)$	$h = 0.1$ $\tau = 0.01$	14.27	0.0466	0.00000	1
2	Explicit scheme	S^2	$h = 0.1$ $\tau = 0.0008$	15.01	0.0334	0.00000	1.11
		$-\frac{4S}{1+S}$	$h = 0.1$ $\tau = 0.0005$	25.89	0.0471	0.00000	1.87
		$\ln(1+S)$	$h = 0.1$ $\tau = 0.0005$	26.94	0.0419	0.00000	1.74
3	Hopscotch scheme I	S^2	$h = 0.08$ $\tau = 0.001$	14.78	0.0472	0.00000	1.09
		$-\frac{4S}{1+S}$	$h = 0.1$ $\tau = 0.003$	6.85	0.0499	0.00002	0.49
		$\ln(1+S)$	$h = 0.05$ $\tau = 0.001$	26.82	0.0433	0.00000	1.88
4	Linearized C-N scheme I	S^2	$h = 0.05$ $\tau = 0.0001$				
		$-\frac{4S}{1+S}$	$h = 0.05$ $\tau = 0.0001$		$L_\infty > 0.4$		
		$\ln(1+S)$	$h = 0.05$ $\tau = 0.0001$				
5	Linearized C-N scheme II	S^2	$h = 0.1$ $\tau = 0.02$	2.16	0.0397	0.00001	0.16
		$-\frac{4S}{1+S}$	$h = 0.1$ $\tau = 0.01$	4.47	0.0497	0.00001	0.32
		$\ln(1+S)$	$h = 0.1$ $\tau = 0.01$	4.64	0.0433	0.00000	0.33
6	Hopscotch scheme II	S^2	$h = 0.1$ $\tau = 0.001$	19.01	0.0374	0.00003	1.40
		$-\frac{4S}{1+S}$	$h = 0.08$ $\tau = 0.001$	23.75	0.0376	0.00003	1.71
		$\ln(1+S)$	$h = 0.1$ $\tau = 0.001$	20.12	0.0457	0.00001	1.41
7	Pseudospectral scheme	S^2	$h = 0.625$ $\tau = 0.0005$	8.23	0.0103	0.00005	0.61
		$-\frac{4S}{1+S}$	$h = 0.625$ $\tau = 0.0008$	5.57	0.0131	0.00008	0.40
		$\ln(1+S)$	$h = 0.625$ $\tau = 0.001$	4.93	0.0111	0.00004	0.35
8	Split step Fourier method	S^2	$h = 0.3125$ $\tau = 0.0003$	14.38	0.0432	0.00007	1.06
		$-\frac{4S}{1+S}$	$h = 0.3125$ $\tau = 0.0007$	10.76	0.0275	0.00006	0.78
		$\ln(1+S)$	$h = 0.625$ $\tau = 0.001$	5.06	0.0169	0.00002	0.36

TABLE V

Comparison of the Computing Time Required to Attain an Accuracy $L_\infty < 0.01$ for Computations from $t = 0$ to $t = 5$ for the Model Problem 2.3

No.	Method	$q(S)$	Step size	Time (s)	$Ł_\infty$	$\mathcal{V}_1$	Ratio
1	C-N scheme	S^2	$h = 0.04$	32.85	0.0082	0.00000	1
			$\tau = 0.01$				
		$-\frac{4S}{1+S}$	$h = 0.04$	40.72	0.0091	0.00001	1
			$\tau = 0.008$				
		$\ln(1+S)$	$h = 0.04$	34.11	0.0094	0.00000	1
			$\tau = 0.01$				
2	Explicit scheme	S^2	$h = 0.04$				
			$\tau = 0.00001$				
		$-\frac{4S}{1+S}$	$h = 0.04$		$L_\infty > 0.01$		
			$\tau = 00001$				
		$\ln(1+S)$	$h = 0.04$				
			$\tau = 0.00001$				
3	Hopscotch scheme I	S^2	$h = 0.04$	227.79	0.0090	0.00001	7.01
			$\tau = 0.0002$				
		$-\frac{4S}{1+S}$	$h = 0.03$	154.71	0.0079	0.00001	3.80
			$\tau = 0.0004$				
		$\ln(1+S)$	$h = 0.03$	319.91	0.0074	0.00001	9.38
			$\tau = 0.0002$				
4	Linearized C-N scheme II	S^2	$h = 0.04$	10.65	0.0067	0.00000	0.32
			$\tau = 0.01$				
		$-\frac{4S}{1+S}$	$h = 0.04$	13.10	0.0099	0.00001	0.32
			$\tau = 0.008$				
		$\ln(1+S)$	$h = 0.04$	11.31	0.0086	0.00000	0.33
			$\tau = 0.01$				
5	Hopscotch scheme II	S^2	$h = 0.04$	394.96	0.0070	0.00000	12.15
			$\tau = 0.0002$				
		$-\frac{4S}{1+S}$	$h = 0.04$	403.45	0.0083	0.00000	9.91
			$\tau = 0.0002$				
		$\ln(1+S)$	$h = 0.04$	414.45	0.0081	0.00001	12.15
			$\tau = 0.0002$				
6	Pseudospectral scheme	S^2	$h = 0.3125$	11.69	0.0076	0.00003	0.36
			$\tau = 0.001$				
		$-\frac{4S}{1+S}$	$h = 0.3125$	14.84	0.0045	0.00002	0.36
			$\tau = 0.001$				
		$\ln(1+S)$	$h = 0.3125$	11.36	0.0068	0.00001	0.33
			$\tau = 0.0008$				
7	Split step scheme	S^2	$h = 0.15625$	17.64	0.0094	0.00005	0.54
			$\tau = 0.0007$				
		$-\frac{4S}{1+S}$	$h = 0.15625$	19.81	0.0069	0.00004	0.49
			$\tau = 0.0008$				
		$\ln(1+S)$	$h = 0.3125$	12.78	0.0051	0.00001	0.38
			$\tau = 0.0008$				

two columns indicate the scheme used. The third column shows the space and time steps used in the computation. The fourth column reports the CPU time used to solve the problem to a given accuracy, and the fifth column shows the actual accuracy of the computed solution. Columns six and seven show how well a scheme conserves the invariant quantities (1.4)–(1.6). The last column gives the ratio of the CPU time of a scheme over that of the implicit C-N scheme. It shows how efficient a scheme is compared with the implicit C-N scheme. All calculations are performed on a SUN workstation. First, we use the eight schemes to compute the one-soliton solution and two-soliton solution of the cubic Schrödinger equation. The computational results are given in Tables I and II, where $L_\infty < 0.01$, i.e., $R_\infty < 1\%$, is required. Second, the plane wave with period $L = 2$ is computed and $L_\infty < 0.02$, i.e., $R_\infty < 1\%$, is required, and the corresponding results are given in Table III. Finally, the computational results for the GNLS equation are given in Tables IV and V.

From these numerical results, we can make the following observations:

(1) It follows from Tables I and II that all eight schemes are capable of computing both the one-soliton solution and the two-soliton solution to the cubic Schrödinger equation. One can use larger time step for the implicit C-N Scheme (3.1) and, larger space step for the pseudospectral scheme (3.10) and the split step Fourier method (3.11)–(3.12). For the cubic Schrödinger equation, the hopscotch scheme I (3.3)–(3.5) is very efficient. The split step Fourier method takes the least computing time among all eight schemes only for NLS and one-soliton model. These two linearized C-N schemes (3.6) and (3.7) are better than the implicit C-N scheme (3.1) for NLS.

(2) The implicit C-N scheme (3.1) is a robust algorithm for the GNLS equation. The step sizes and the computing time are not sensitive to the function q.

(3) In general, the pseudospectral scheme (3.10) and the split step Fourier method take less computing time, but they do not keep conservative laws well. In order to attain better accuracy, one has to take a small time step size. The split step Fourier method (3.11)–(3.12) is accurate only when solution varies slowly with time. Otherwise, the method is not efficient, since the accuracy of the formula (3.11) depends on the term $q(|u(x,t)|^2)$, in general.

(4) Overall, the linearized C-N scheme II (3.7) is the most efficient of the eight schemes. Especially for the model problem (2.3), the one-soliton solution of the GNLS equation, the ratio of the computing time is less than 0.33, because the gradient of the solution is larger than the gradients of problems (2.1) and (2.2). The numerical results confirm our analysis in Section 4 for the linearized C-N scheme II.

(5) The explicit scheme (3.2) may be used to compute the GNLS equation if high accuracy is not required. According to linearized stability analysis, the scheme (3.2) is stable if $\gamma = \tau/h^2 \le 1/4$. In the computations, we see that the scheme (3.2) with the time step size $\tau = h^2/4$ works well only for the cubic Schrödinger equation. For the model problem (2.2), the time step size τ should be strictly less than $h^2/4$ for stability. The numerical results for the model problem (2.3) show that the scheme is unstable for $h = 0.1$, $\tau = 0.001$.

(6) The hopscotch scheme I (3.3)–(3.5) is very sensitive to the solution and the nonlinear term $q(s)$. It is the most efficient scheme for the cubic Schrödinger equation, but takes the most computing time for problem (2.2) with $q(s) = s^2$ or $q(s) = \ln(1+s)$. In addition, the computing time for solving the problem with $q(s) = -4s/(1+s)$ is much less than that for $q(s) = s^2$ or $q(s) = \ln(1+s)$. It can be observed from Tables I–IV that the condition $\tau < h^2$ is necessary. This observation is consistent with our analysis that the truncation error of this scheme is of order $O(\tau^2 + h^2 + \tau^2/h^2)$.

(7) The linearized C-N scheme (3.6) can only be used to compute the cubic Schrödinger equation and is unsatisfactory when applied to the GNLS equation. Our numerical results show that scheme (3.6) is stable and the solution is bounded as time increases. However, for the GNLS equation, the computational error L_∞ is always too large to satisfy the accuracy requirement no matter how small h and τ are taken. This observation can be explained as follows. Because the scheme has a phase error of $O(\tau)$, a very small τ will be needed to reduce this phase error to the given tolerance due to the low accuracy of the scheme. Therefore, a large number of time steps are needed to compute the solution in a given time T, and round-off error becomes dominant so that the computed error will always remain above tolerance.

(8) The hopscotch scheme with extrapolation (3.8)–(3.9) can be used to compute all three-model problems, but it takes more computing time when solving the GNLS equations.

We have presented a new linearized C-N-type finite difference scheme based on the use of an extrapolation technique and compared with other seven existing finite difference schemes by examining three-model problems of a nonlinear generalized Schrödinger equation. It is proved that the scheme satisfies a basic conservation law and is of good dissipation and dispersion. Our numerical experiments demonstrate that the linearized C-N scheme II (3.7) is most efficient and robust in general for solving the GNLS equation. Some other schemes can be efficient for some special model problems. Numerical observations and suggestions have been made, which may be helpful in choosing a suitable scheme for a special case. Since the underlying GNLS equation contains a complicated nonlinear term which often appears in many other physical equations, our work will provide some useful information for solving other nonlinear partial differential equations.

ACKNOWLEDGMENT

The authors thank the referees for their valuable comments which led to the improvement of this paper.

REFERENCES

1. I. Blalynickl-Birdla and J. Mycielski, Gaussons: Solutions of the logarithmic Schrödinger equation, *Phys. Scripta* **20**, 539 (1979).
2. R. T. Bullough, P. M. Jack, P. W. Kitchenside, and R. Saunders, Solutions in laser physics, *Phys. Scripta* **20** 364 (1979).
3. Q. Chang, Conservative difference scheme for generalized nonlinear Schrödinger equations, *Sci. Sinica Ser. A* **26**, 687 (1983).
4. Q. Chang and G. Wang, Multigrid and adaptive algorithm for solving the nonlinear Schrödinger equations, *J. Comput. Phys.* **88**, 362 (1990).
5. Q. Chang and L. Xu, A numerical method for a system of generalized nonlinear Schrödinger equations, *J. Comput. Math.* **4**, 191 (1986).
6. S. Cowan, R. H. Enns, S. S. Rangnekar, and S. S. Sanghera, Quasi-soliton and other behaviour of the nonlinear cubic-quintic Schrödinger equation, *Canad. J. Phys.* **64**, 311 (1986).
7. K. Konno and H. Suzuki, Self-focussing of laser beams in nonlinear media, *Phys. Scripta* **20**, 382 (1979).
8. D. Pathria and J. L. Morris, Pseudo-spectral solution of nonlinear Schrödinger equations, *J. Comput. Phys.* **87**, 108 (1990).
9. W. Strauss, Mathematical aspects of classical nonlinear field equations, *Lect. Notes Phys.* **98**, 123 (1979).

10. C. Sulem, P. L. Sulem, and A. Patera, Numerical simulation of singular solutions to the two-dimensional cubic Schrödinger equation, *Comm. Pure Appl. Math.* **37**, 755 (1984).
11. T. R. Taha and M. J. Ablowitz, Analytical and numerical aspects of certain nonlinear evolution equations. II. numerical, nonlinear Schrödinger equation, *J. Comput. Phys.* **55**, 203 (1984).
12. B. Fornberg and G. B. Whitham, *Philos. Trans. Roy. Soc.* **289**, 373 (1978).
13. D. Pathria and J. L. Morris, Exact solutions for generalized nonlinear Schrödinger equation, *Phys. Scripta* **39**, 673 (1989).